Introduction to Computer Graphics

Spring 2011, lecture notes #2

Mathematics background

02.02.2011

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- Abstractions
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- Curves: lines, parametric, implicit
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Abstractions

- The mathematical foundations of computer graphics are based on infinite precision abstractions of geometrical objects.
- For instance, a **point** in space is 0-dimensional with zero length, zero area, zero volume, ... while a **line** has non-zero length but zero area, zero volume, and so on.
- Care should be taken when calculating positions, intersection points, etc., using finite precision arithmetic.

Trigonometry

- Trigonometric functions are used both directly and indirectly. All angles are measured in *radians.*
- Direct use
  - Rotations by an angle θ are expressed using $\sin(\theta)$ and $\cos(\theta)$.
  - Physical models of the behaviour of light at material interfaces use $\sin(\phi)$ and $\cos(\phi)$, needed to calculate photorealistic reflections and refractions.
- Indirect use
  - To decide if two vectors are pointing in the same or opposite directions: cos > 0 or < 0 .

Trigonometry

- $\sin(-\alpha) = -\sin(\alpha)$, but $\cos(-\alpha) = \cos(\alpha)$
- Addition formulae

  $\sin(\alpha+\beta) = \sin(\alpha)*\cos(\beta) + \cos(\alpha)*\sin(\beta)$

  $\cos(\alpha+\beta) = \cos(\alpha)*\cos(\beta) - \sin(\alpha)*\sin(\beta)$
- Subtraction formulae

  $\sin(\alpha-\beta) = \sin(\alpha)*\cos(\beta) - \cos(\alpha)*\sin(\beta)$

  $\cos(\alpha-\beta) = \cos(\alpha)*\cos(\beta) + \sin(\alpha)*\sin(\beta)$

Points

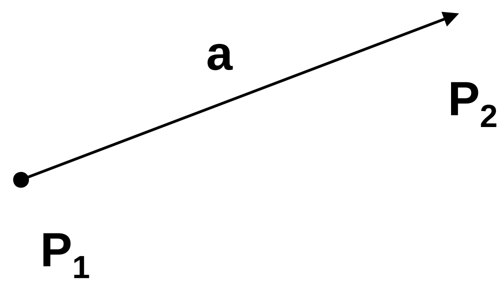
- Points in n-dimensional Euclidean spaces are given by n-tuples

$$P = (x_1, x_2, .., x_n)$$

  We will mostly have $n \leq 4$.
- Points are used to define straight lines and curves.
- Straight lines are used to define planes and areas. Using points and eg. splines or paramteric representations we can construct curved surfaces.
- Planes are used to define volumes.

Vectors

- A vector **a** has a direction and a length
- The vector **a** can be defined by giving its starting point P_1 and end point P_2.

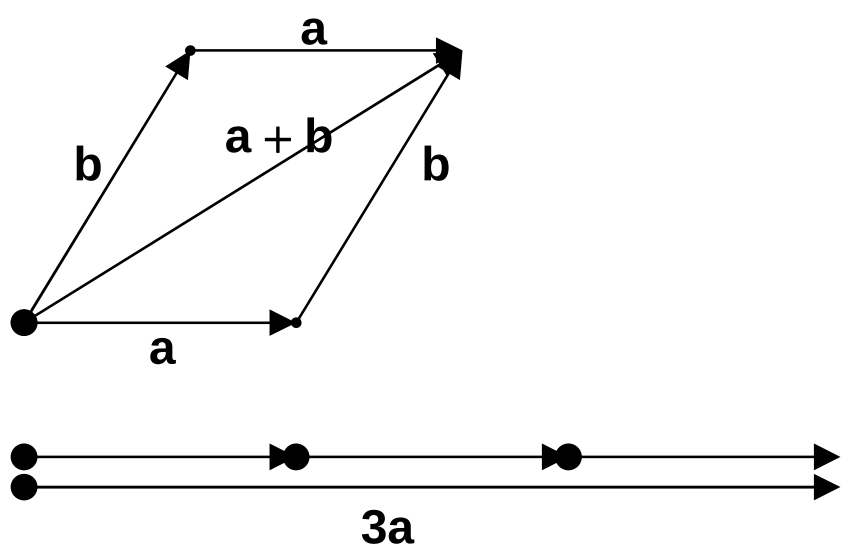


- A vector is not fixed in space. You may freely translate it preserving its direction.

Vectors

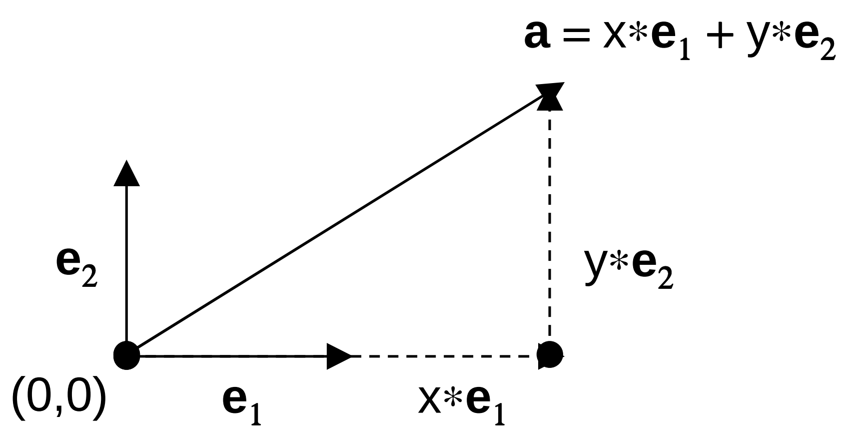
- Vector algebra: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

$$\alpha * \mathbf{a} = \alpha\mathbf{a}$$



Vectors

- Unit basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in a 2D Cartesian coordinate system:



- Unit length: $||\mathbf{e}_1|| = ||\mathbf{e}_2|| = 1$
- If P_1 and P_2 themselves are vectors defined from the origin (0,0,..,0) then $\mathbf{a} = P_2 - P_1$

Vectors

- Any vector **a** can be represented by its unique coordinates (x,y) in a Cartesian coordinate system: $\mathbf{a} = x * \mathbf{e}_1 + y * \mathbf{e}_2$
- Coordinate representation:

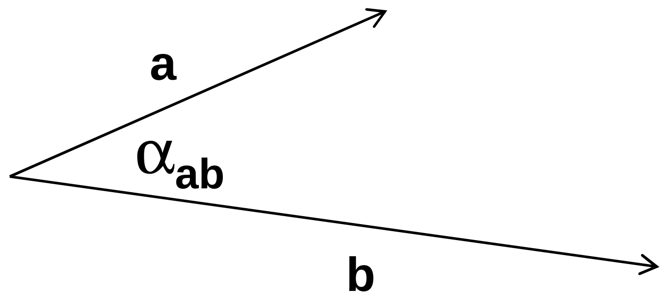
  $\mathbf{a}^T = [x , y]$
- $||\mathbf{a}|| = \mathrm{sqrt}(x^2 + y^2)$, Pythagoras

Dot product $\mathbf{a} \cdot \mathbf{b}$

- For any two vectors **a** and **b** we can calculate their dot product $\cdot$

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| * ||\mathbf{b}|| * \cos(\alpha_{\mathbf{ab}})$$

where α_{ab} = the angle between **a** and **b**



- In the coordinate representation

$$\mathbf{a} \cdot \mathbf{b} = x_a * x_b + y_a * y_b$$

Dot product **a** · **b**

- Combining these two results we have a way of calculating the cosine of the angle between two vectors using only square roots and multiplications:

  $\cos(\alpha_{\mathbf{ab}}) = (x_a * x_b + y_a * y_b)/(||\mathbf{a}|| * ||\mathbf{b}||)$

- If **a** and **b** are perpendicular to each other then $\mathbf{a} \cdot \mathbf{b} = 0$.

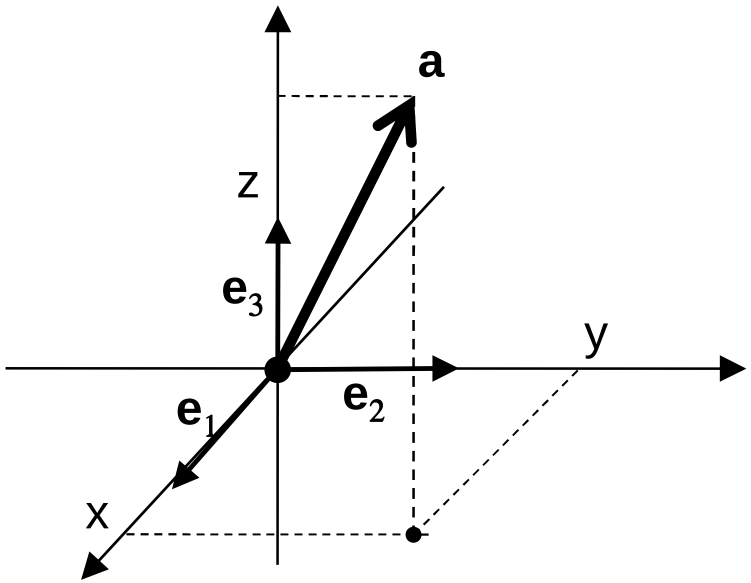
Dot product $\mathbf{a} \cdot \mathbf{b}$

- In 3D

  $\mathbf{e}_1 = (1,0,0)$

  $\mathbf{e}_2 = (0,1,0)$

  $\mathbf{e}_3 = (0,0,1)$



- $\mathbf{a} = x*\mathbf{e}_1 + y*\mathbf{e}_2 + z*\mathbf{e}_3$
- $||\mathbf{a}|| = \mathrm{sqrt}(x^2 + y^2 + z^2)$
- $\mathbf{a} \cdot \mathbf{b} = x_a * x_b + y_a * y_b + z_a * z_b$

Cross product **a x b**

- Defined in 3D for two vectors **a** and **b**

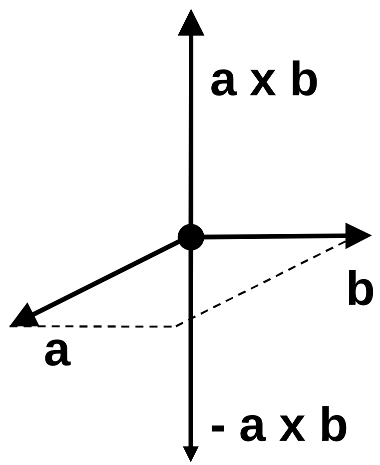
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$(\mathbf{a} \times \mathbf{b})_x = \quad a_y * b_z - a_z * b_y$$
$$(\mathbf{a} \times \mathbf{b})_y = - a_x * b_z + a_z * b_x$$
$$(\mathbf{a} \times \mathbf{b})_z = \quad a_x * b_y - a_y * b_x$$

- Also $||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}|| * ||\mathbf{b}|| * \sin(\alpha_{ab})$

Cross product **a x b**

- **a** x **b** = **0** iff **a** and **b** are parallel, **a** x **a** = 0
- **a** x **b** = – **b** x **a**
- If **a** and **b** are not parallel to each other then **a** x **b** will be perpendicular to the plane spanned by **a** and **b**.



Local coordinate basis

- An orthonormal basis of three unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ perpendicular to each other can be formed from any three vectors **a**, **b**, **c** not collinear by the following procedure:
  - $\mathbf{e}_1$ = normalized **a**: $\mathbf{e}_1 = \mathbf{a}/||\mathbf{a}||$
  - Form $\mathbf{b}' = \alpha\,\mathbf{e}_1 + \beta\,\mathbf{b}$ such that $\mathbf{b}'\cdot\mathbf{e}_1 = 0$, that is, $\alpha + \beta\,\mathbf{b}\cdot\mathbf{e}_1 = 0$
    - If $\mathbf{b}\cdot\mathbf{e}_1 = 0$ choose $\alpha = 0$ and $\mathbf{e}_2 = \mathbf{b}/||\mathbf{b}||$
    - Else let $\beta = 1$, $\alpha = -\mathbf{b}\cdot\mathbf{e}_1$, and $\mathbf{e}_2 = \mathbf{b}'/||\mathbf{b}'||$
  - $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$

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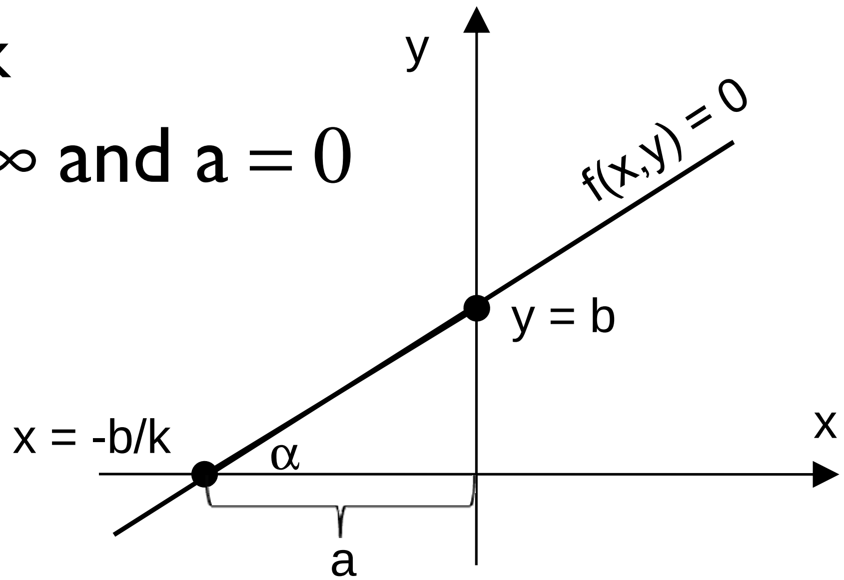
Curves, lines

- Curves are 1D objects with non-zero length but zero area, zero volume, etc.
- We consider only curves in Cartesian coordinate spaces, 2D or 3D.
- Can be defined
  – *implicitly* from an equation $f(x,y) = 0$
  eg. $f(x,y) = y - k*x - b$
  – *parametrically*, $[x,y] = [f(t),g(t)], \quad t \in [0,1]$

Straight line in 2D, implicit

- Definition: $f(x,y) = y - k*x - b$
- Points on the line given by $f(x,y) = 0$
  - k is the slope, $k = \tan(\alpha) = b/a$
  - $x = 0 \Rightarrow y = b$, the y-intercept
  - $y = 0 \Rightarrow x = -b/k$
- Special cases: $k = \infty$ and $a = 0$

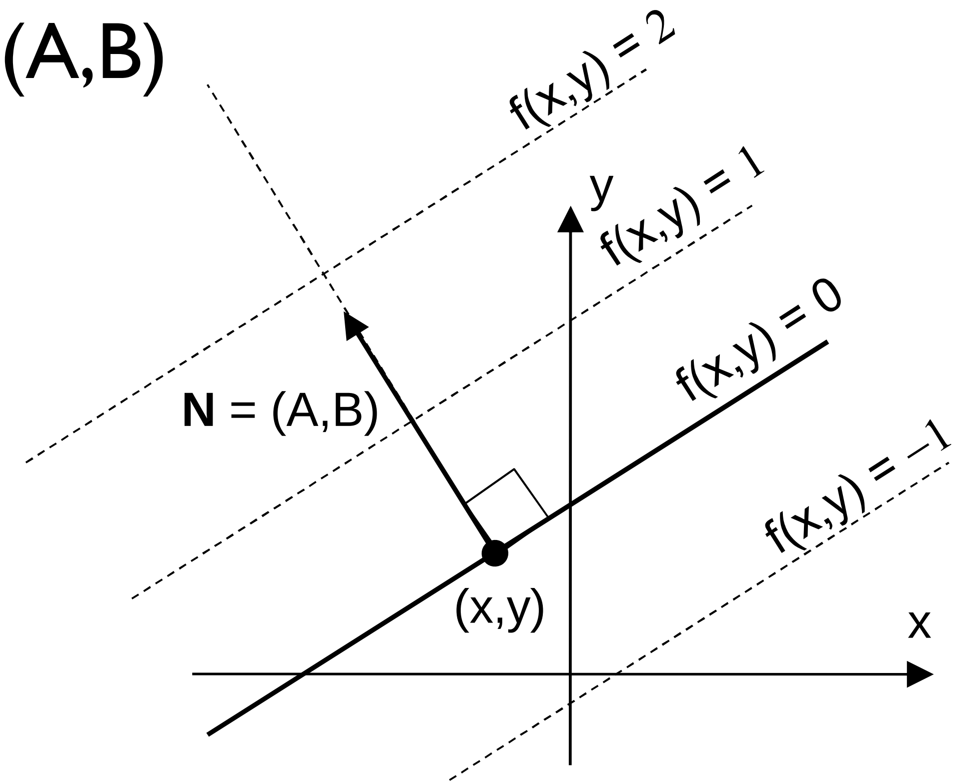


Straight line in 2D, implicit

- A more general definition:

  $$f(x,y) = A*x + B*y + C$$

- Normal: (A,B)



Straight line in 2D, implicit

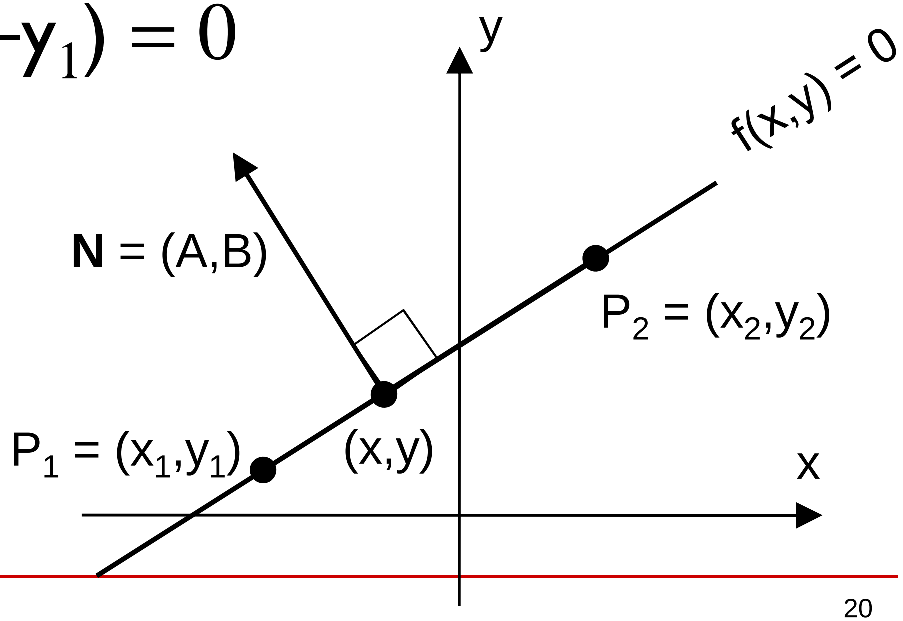
- Given two points on the line, find A,B,C!
- A,B,C not unique: we can multiply by any non-zero value.
- The vector $\mathbf{P}_2 - \mathbf{P}_1$ is orthogonal to $\mathbf{N}$:

  $(A,B) \cdot (x_2 - x_1, y_2 - y_1) = 0$

  choose

  $A = y_1 - y_2$

  $B = x_2 - x_1$



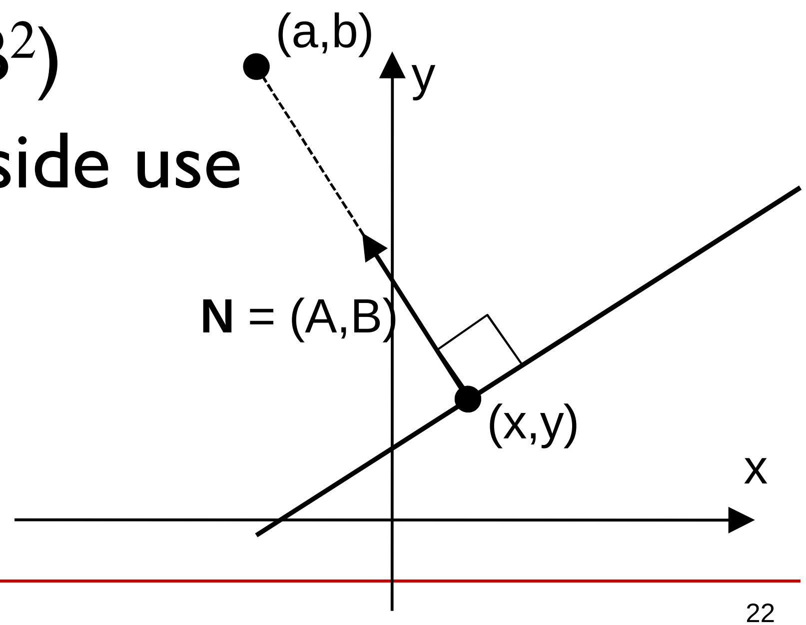
Straight line in 2D, implicit

- Thus $(y_1\text{-}y_2)*x + (x_2\text{-}x_1)*y + C = 0$
- This line also passes through $(x,y) = P_1$. Use $x = x_1$ and $y = y_1$ and solve for C.

$$(y_1\text{-}y_2)*x + (x_2\text{-}x_1)*y + x_1y_2 - x_2y_1 = 0$$

Straight line in 2D, implicit

- Distance d from (a,b) to lin, at (x,y)?
- d is along normal, $d = k*\text{sqrt}(A^2+B^2)$
- $(a,b) = (x,y) + k(A,B)$, calculate
  $f(a,b) = Ax+kA^2+By+kB^2+C = k(A^2+B^2)$
- $d = f(a,b)/\text{sqrt}(A^2+B^2)$
- if (a,b) on negative side use $-d$



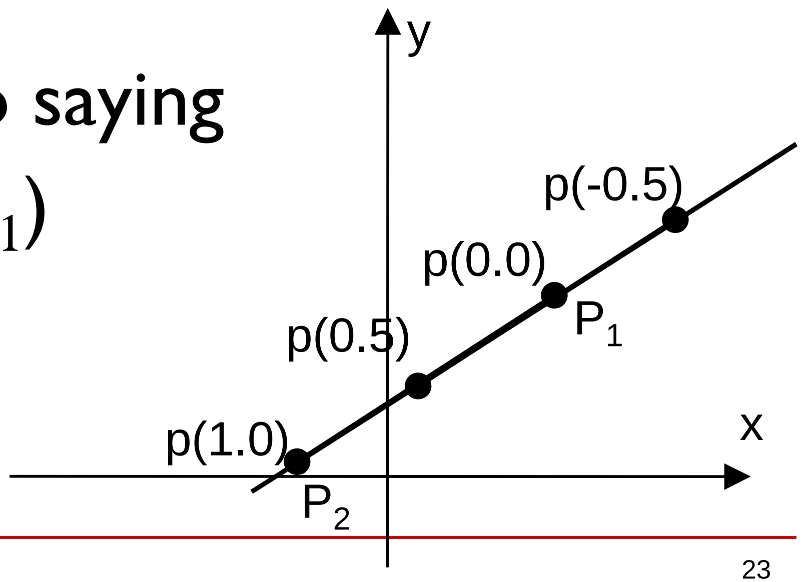
Straight line in 2D, parametric

- The line passing through $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ can be written

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 + t(x_2-x_1) \\ y_1 + t(y_2-y_1) \end{bmatrix} \qquad t \in (-\infty, +\infty)$$

- This is equivalent to saying

$$p(t) = P_1 + t*(P_2 - P_1)$$



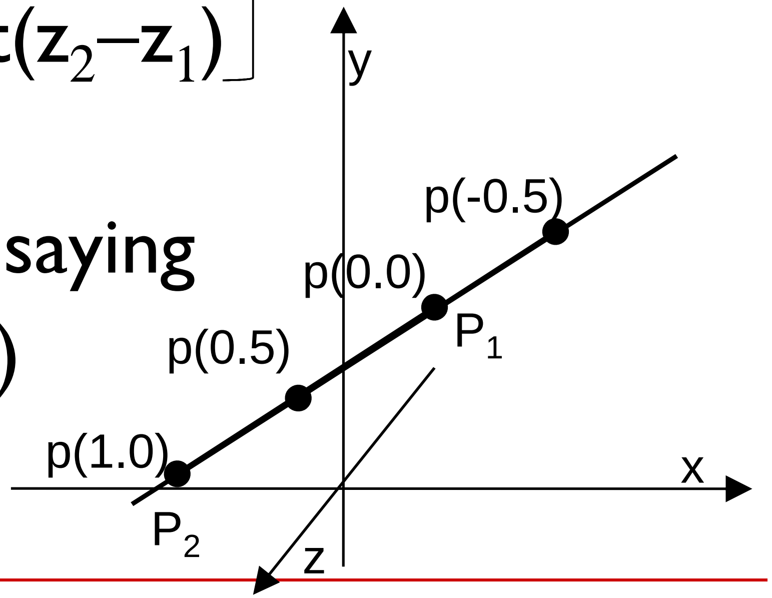
Straight line in 3D, parametric

- The line passing through $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ can be written

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 + t(x_2 - x_1) \\ y_1 + t(y_2 - y_1) \\ z_1 + t(z_2 - z_1) \end{bmatrix} \qquad t \in (-\infty, +\infty)$$

- This is equivalent to saying

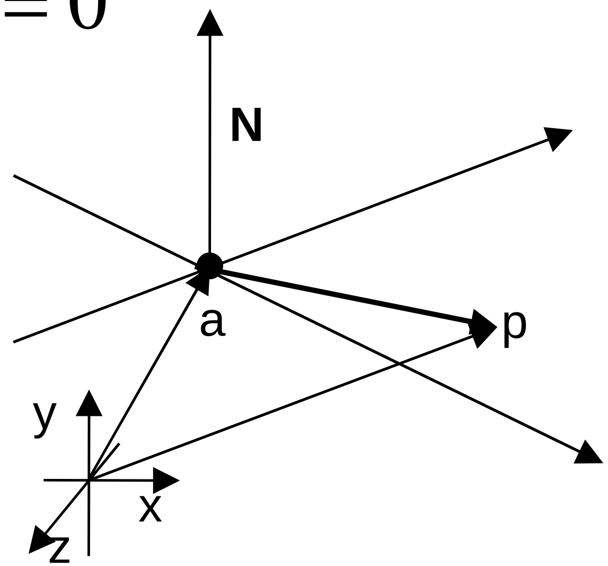
$$p(t) = P_1 + t * (P_2 - P_1)$$



Plane in 3D, implicit

- Given the point **a** and a normal N, can we formulate an equation for all points in a plane for which N is the normal? Denote any point in the plane by **p**.

$$(\mathbf{p} - \mathbf{a}) \cdot \mathbf{N} = 0$$

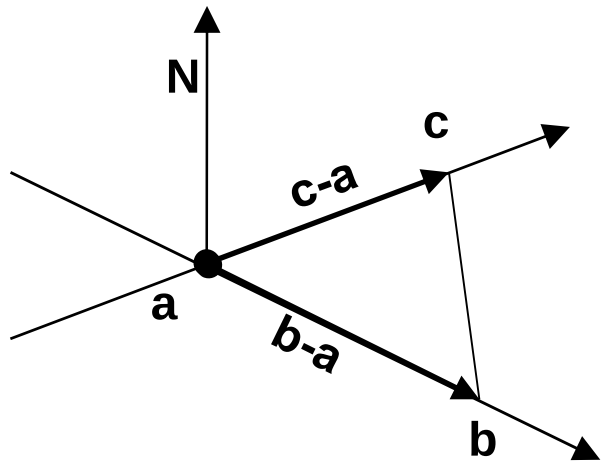


Plane in 3D, implicit

- Three points **a,b,c** define a plane. Can we formulate an equation for this plane?

$$(\mathbf{p} - \mathbf{a}) \cdot \mathbf{N} = 0$$

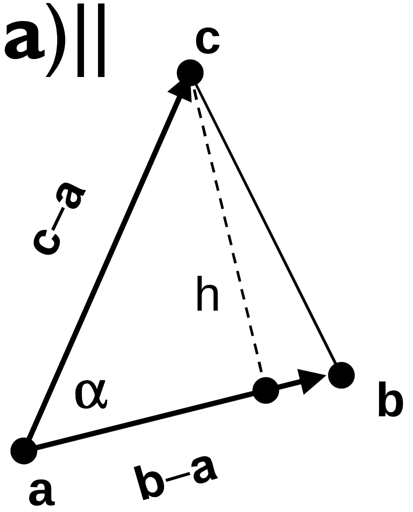
$$\mathbf{N} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$$



Triangles, 2D

- Three points **a,b,c** in 2D define a triangle.
- Its area $A = \text{base} * \text{height}/2 = ||\mathbf{b} - \mathbf{a}|| * h/2$
- But $\sin(\alpha) = h/||\mathbf{c} - \mathbf{a}||$ therefore

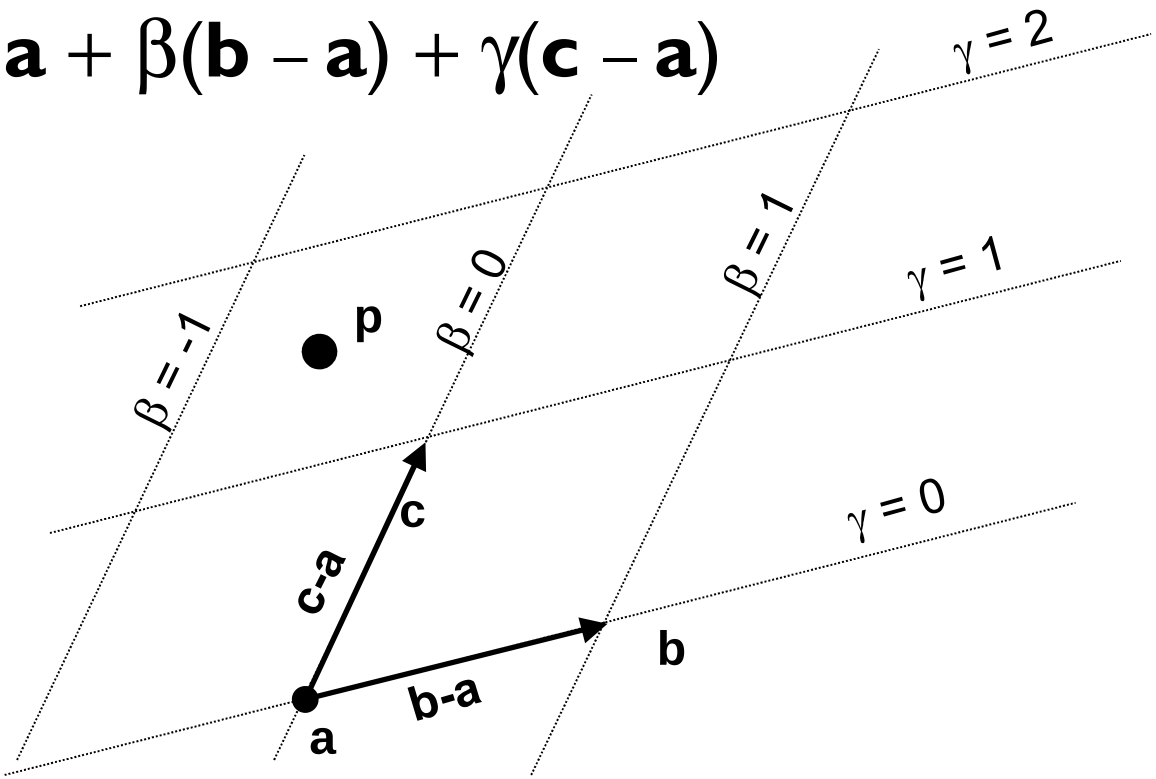
$$A = \tfrac{1}{2}||\mathbf{b} - \mathbf{a}|| * ||\mathbf{c} - \mathbf{a}|| * \sin(\alpha)$$
$$= \tfrac{1}{2}\,||(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})||$$



Triangles, barycentric coordinates

- Any point **p** in the plane defined by the triangle **abc** can be written

$$\mathbf{p} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$$



Triangles, barycentric coordinates

- Rewrite as

$$\mathbf{p} = (1 - \beta - \gamma)\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

- Define $\alpha = (1 - \beta - \gamma)$
- Now

$$\mathbf{p} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} \qquad \alpha + \beta + \gamma = 1$$

- P is inside the triangle provided

$0 < \alpha < 1$ and $0 < \beta < 1$ and $0 < \gamma < 1$

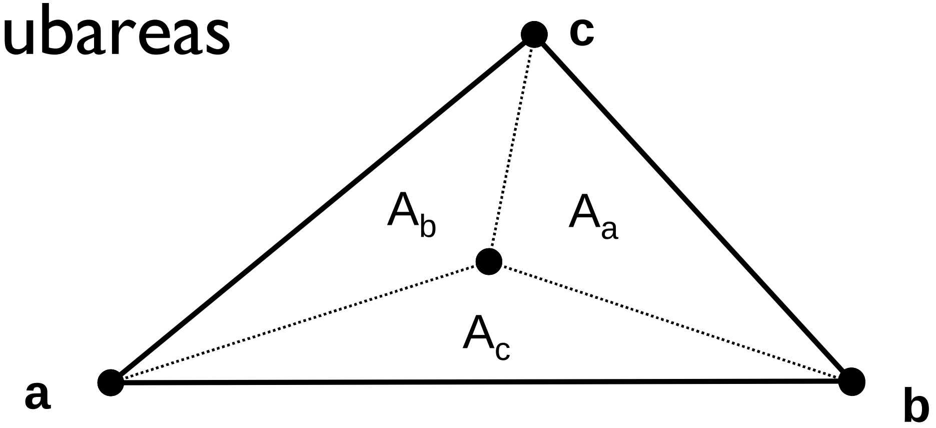
Triangles, barycentric coordinates

- The barycentric coordinates b.c. give a smooth interpolation of values given at the vertices of the triangle:
  - if two of the b.c. are zero we are at one of the vertices, eg. $\alpha = 1, \beta = \gamma = 0$ at **a**
  - the b.c. coordiantes are proportional to triangle subareas

$\alpha = A_a/A$

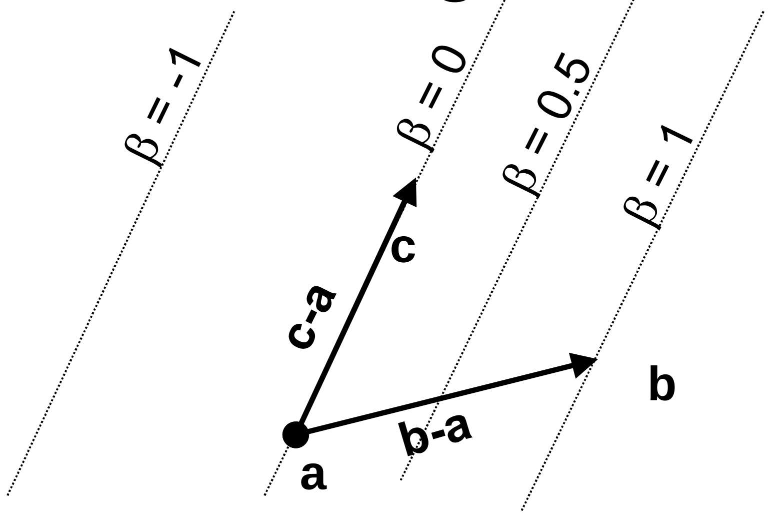
$\beta = A_b/A$

$\gamma = A_c/A$



Triangles, barycentric coordinates

- Given **a**,**b**,**c** how do we calculate the barycentric coordinates α,β,γ?
- We already saw that the implicit form f(x,y) = Ax + By +C of a line gives a value proportional to the signed distance to the line



Triangles, barycentric coordinates

- Therefore, for any point (x,y)

$$\beta = f_{ac}(x,y)/f_{ac}(x_b,y_b)$$
$$\gamma = f_{ab}(x,y)/f_{ab}(x_c,y_c)$$
$$\alpha = 1 - \beta - \gamma$$

- In 3D, all formulae for the barycentric coordinates are the same, only add the third coordinate to the points **a**,**b**,**c**!

Triangles in 3D, barycentric c.

- In 3D, any point **p** in the plane of the triangle with vertices **a**,**b**,**c** is given by

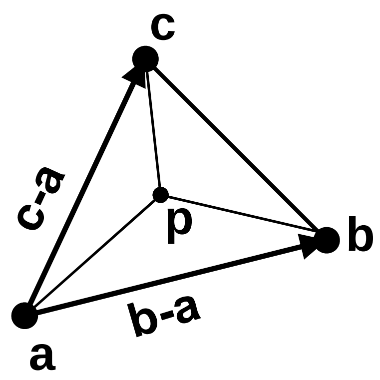
$$\mathbf{p} = (1 - \beta - \gamma)\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

- The length of the normal

$$\mathbf{n} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$$

  is proportional to the triangle area: $A = |\mathbf{n}|/2$

- Idea: Let's calculate the subtriangle areas formed by **p** and the triangle vertices **a,b,c** !



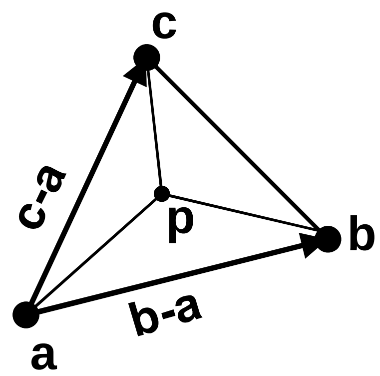
Triangles in 3D, barycentric c.

- We should make sure the vertices of the subtriangle form a right-handed coordinate system when calculating $\mathbf{n}_i$:

$$\mathbf{n}_a = (\mathbf{c} - \mathbf{b}) \times (\mathbf{p} - \mathbf{b}) \rightarrow \alpha = \mathbf{n} \cdot \mathbf{n}_a / |n|^2$$

$$\mathbf{n}_b = (\mathbf{a} - \mathbf{c}) \times (\mathbf{p} - \mathbf{c}) \rightarrow \beta = \mathbf{n} \cdot \mathbf{n}_b / |n|^2$$

$$\mathbf{n}_c = (\mathbf{b} - \mathbf{a}) \times (\mathbf{p} - \mathbf{a}) \rightarrow \gamma = \mathbf{n} \cdot \mathbf{n}_c / |n|^2$$



Matrices

- Let A,B and C be three matrices of the same size, preferably square!
- Basic matrix algebra
  - Unit matrix I : ones on the diagonal, zero otherwise, $I_{ij} = \delta_{ij}$, δ_{ij} Kronecker delta
  - Addition: $A + B = B + A$, $(A + B)_{ij} = A_{ij}+B_{ij}$
  - Subtraction: $A - B$, $(A - B)_{ij} = A_{ij}-B_{ij}$
  - Multiplication by constant α : $(\alpha A)_{ij} = \alpha A_{ij}$
  - Generally $A \cdot B \neq B \cdot A$
  - Distributivity: $A \cdot (B+C) = A \cdot B + A \cdot C$

Matrices

- Inverse matrix

  A^{-1} s.t. $A^{-1} \cdot A = A \cdot A^{-1} = I$

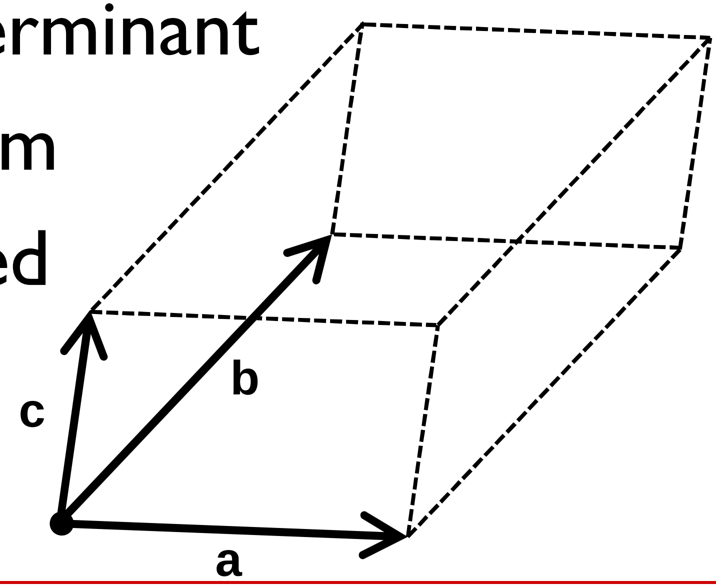
  $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$
- Matrix transpose $A^T_{ij} = A_{ji}$, $(A \cdot B)^T = B^T \cdot A^T$
- A is *symmetric* if $A^T = A$
- U is *orthogonal* if its columns consists of vectors which have length 1 and are orthogonal to each other. Then the rows are also orthogonal, and $A \cdot A^T = A^T \cdot A = I$.

Matrices

- The inverse matrix exists only if the determinant $|A| \neq 0$.
- $|AB| = |A||B|$, $|A^T| = |A|$
- $|A^{-1}| = 1/|A|$
- For U orthogonal, $|U| = \pm 1$
- Solving equations: if $A \cdot \mathbf{v} = 0$ then nontrivial $\mathbf{v}$ exists only if $|A| = 0$.

Volume = ABS(Determinant)

- The volume of the parallelepiped formed by three vectors **a**,**b**,**c** is the absolute value of the determinant |**abc**| where the components of the vectors are row or columns in the determinant
- Note! Do **a,b,c** form a right or left handed coordinate system?



Eigenvalues

- nxn square matrices A have n eigenvalues λ and corresponding n eigenvectors **v** s.t.

  $$A \cdot \mathbf{v} = \lambda \, \mathbf{v}$$

- They are found by calculating

  $$(A - \lambda I) \cdot \mathbf{v} = 0 \Leftrightarrow |A - \lambda I| = 0$$

  This gives an n-th order polynomial in λ with n solutions.

- Eg. in 3D, for real λ, **v** respresents a direction in which A is *invariant*.

Singular Value Decomposition

- If A is symmetric, the eigenvalues are real, and $A = Q \cdot D \cdot Q^T$ where D is a diagonal with eigenvalues λ_i and Q has columns with corresponding eigenvectors.
- Any square matrix A can be decomposed

  $$A = U \cdot S \cdot V^T$$

  where S is diagonal with non-negative singular values σ_i of A, and U,V are orthogonal. If $A^T = A$ then $\sigma_i = \mathrm{sqrt}(\lambda_i * \lambda_i)$

Complex Numbers

- Introduced to extend real numbers to include an imginary number $\mathbf{i}$ such that $\mathbf{i}^2 = -1$. Symbolically $\mathbf{i}$ = sqrt(−1)
- Any complex number *z* can be written using two real numebrs *x,y* and the imaginary unit: $z = x + \mathbf{i} * y$
- Complex numbers appear often as roots for polynomials.
- Functions f(z) have strong properties.

Quaternions

- Quaternions were invented as an extension of complex numbers to a set of elements with an associative division algebra.
- Idea: try w = $x + \mathbf{i} * y + \mathbf{j} * z$, x,y,z real, $\mathbf{i}^2 = \mathbf{j}^2 = -1$. Unfortunately nobody has been able to make any sense of $\mathbf{i} * \mathbf{j}$

Quaternions

- Quaternion numbers q are given by

  $$q = s+a*\mathbf{i}+b*\mathbf{j}+c*\mathbf{k} = (s,a,b,c) = (s,\underline{\mathbf{v}})$$

- $s,a,b,c \in \Re$
- i,j,k satisfy the algebra:

  $$\mathbf{i}*\mathbf{i} = \mathbf{j}*\mathbf{j} = \mathbf{k}*\mathbf{k} = -1$$
  $$\mathbf{i}*\mathbf{j} = -\mathbf{j}*\mathbf{i} = \mathbf{k}$$
  $$\mathbf{j}*\mathbf{k} = -\mathbf{k}*\mathbf{j} = \mathbf{i}$$
  $$\mathbf{k}*\mathbf{i} = -\mathbf{i}*\mathbf{k} = \mathbf{j}$$

- Quaternions are not commutative.

Quaternions

- Algebra:

  $|q|^2 = s^2 + a^2 + b^2 + c^2 = s^2 + \underline{v} \cdot \underline{v}$

  $q^{-1} * q = (1, \underline{0})$

  $q^{-1} = (s, -\underline{v}) / |q|^2$

- Quaternion multiplication:

  $q_1 = (s_1, \underline{\mathbf{v}}_1), \quad q_2 = (s_2, \underline{\mathbf{v}}_2)$

  $q_1 * q_2 = (s_1 s_2 - \underline{\mathbf{v}}_1 \cdot \underline{\mathbf{v}}_2,\ s_1 \underline{\mathbf{v}}_2 + s_2 \underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_1 \times \underline{\mathbf{v}}_2)$

Quaternions

- Rotations in 3D are usually formulated using 3x3 or 4x4 matrices.
- Quaternions can be used to formulate a 2x2 matrix representation of rotations in three dimensions.
- Both representations have their advantages and disadvantages